Problem 1 (25 points)

Each of the following is sampled at $F_s = 10000$ samples/second, producing either $x[n] =$constant, or $x[n] = \cos \omega n$ for some value of ω. Specify the constant if possible; otherwise, specify ω such that $-\pi \leq \omega < \pi$.

(a) $x(t) = \cos(2\pi 900 t)$

Solution: $\omega = \frac{9\pi}{50}$

(b) $x(t) = \cos(2\pi 10000 t)$

Solution: $x[n] = 1$

(c) $x(t) = \cos(2\pi 11000 t)$

Solution: $\omega = \frac{\pi}{5}$

Problem 2 (25 points)

Consider the signal

$$x(t) = 2\cos(2\pi 440 t) - 3\sin(2\pi 440 t)$$

This signal can also be written as $x(t) = A\cos(\omega t + \theta)$ for some $A = \sqrt{M}$, ω, and $\theta = \text{atan}(R)$. Find M, ω, and R.

Solution:

$$\begin{aligned} A &= \sqrt{5} \quad (M = 5) \\ \omega &= 2\pi 440 \\ \theta &= \text{atan}\left(\frac{3}{2}\right) \quad (R = \frac{3}{2}) \end{aligned}$$

Problem 3 (25 points)

A signal $x(t)$ is periodic with $T_0 = 0.02$ seconds, and its values are specified by

$$x(t) = \begin{cases} -1 & 0 \leq t \leq 0.01 \\ 0 & 0.01 < t < 0.02 \end{cases}$$

Its CTFS representation is defined by

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}$$

(a) Sketch $x(t)$ as a function of t for $0 \leq t \leq 0.02$ seconds. Label at least one important tic mark, each, on the horizontal and vertical axes.

Solution: Useful tic marks include $t = 0.01$ or $t = 0.02$, and $x(t) = -1$ between 0 and 0.01.

(b) What is ω_0?

Solution: $\omega_0 = 100\pi$

(c) Find X_0 without doing any integral.

Solution: $X_0 = -\frac{1}{2}$

(d) Find X_k for all the other values of k, i.e., for $k \neq 0$. Simplify; your answer should have no exponentials in it.

Solution: $X_k = 0$ for even k, $X_k = -\frac{j}{k\pi}$ for odd k.

Problem 4 (25 points)

Consider the signal

$$x[n] = \begin{cases} \left(\frac{1}{2}\right)^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

(a) Find the DTFT, $X(\omega)$.

Solution: $X(\omega) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$

(b) Find the power spectrum $|X(\omega)|^2$, and sketch it for $-\pi \leq \omega \leq \pi$. Specify its values at $\omega = 0$, $\omega = \frac{\pi}{2}$, and $\omega = \pi$.

Solution: $|X(\omega)|^2 = \frac{1}{\frac{5}{4}-\cos\omega}$. $|X(0)|^2 = 4$, $|X(\frac{\pi}{2})|^2 = \frac{4}{5}$, $|X(\pi)|^2 = \frac{4}{9}$.