ECE 401 Signal and Image Analysis
Homework 2

UNIVERSITY OF ILLINOIS
Department of Electrical and Computer Engineering

Assigned: Tuesday, 9/1/2020; Due: Monday, 9/14/2020
Reading: *DSP First* pp. 12-34, 50-58, 61-71

Problem 2.1

In standard tuning, the middle A note on a piano (A4) has a frequency of 440Hz. Consider the note

$$x(t) = 14\cos\left(2\pi 440t + 0.88\pi\right)$$

Sketch one complete period of $x(t)$, from its first peak after $t=0$ until its second peak after $t=0$. Label the times of both peaks, and the value of $x(t)$ at both peaks.

Solution: The peaks are at

$$440(t + 0.001) = k \ldots$$

where k is any integer. The first value of k that gives a positive t is $k=1$,

$$t = \frac{1}{440} - 0.001 \approx 0.0013$$

The second peak is at

$$t = \frac{2}{440} - 0.001 \approx 0.0035$$

The amplitude is $A = 14$.

Problem 2.2

Suppose you're given the signal

$$x(t) = \cos(2\pi 440t) + 3\sin(2\pi 440t)$$

Find the phasor representation of $x(t)$, and simplify it to polar form. You might want to take advantage of facts like $\sin(x) = \cos(x - \frac{\pi}{2})$, and $\sin(\frac{\pi}{2}) = 1$, and $\cos(\frac{\pi}{2}) = 0$.

Solution:

$$\begin{aligned} x(t) &= \cos(2\pi 440t) + 3\cos\left(2\pi 440t - \frac{\pi}{2}\right) \\ &= \Re\left\{e^{j2\pi 440t} + 3e^{j2\pi 440t}e^{-j\frac{\pi}{2}}\right\} \end{aligned}$$

So the phasor is

$$\begin{aligned} 1 + 3e^{-j\frac{\pi}{2}} &= 1 + 3\cos(\pi/2) - 3j\sin(\pi/2) \\ &= 1 - 3j \\ &= \sqrt{10}e^{-j\text{atan}(3)} \end{aligned}$$

Problem 2.3

Kwikwag's beat-tones example on Wikipedia adds two tones, at the frequencies 110Hz and 104Hz:

$$x(t) = \cos(2\pi 110 t) + \cos(2\pi 104 t)$$

Find a sequence of frequencies and phasors, $\{(f_{-2}, a_{-2}), \ldots, (f_2, a_2)\}$, such that

$$x(t) = \sum_{k=-2}^{2} a_k e^{j2\pi f_k t}$$

Solution: The easiest way to solve this one is to just use Euler's identity:

$$x(t) = \frac{1}{2}\left(e^{j2\pi 110t} + e^{-j2\pi 110t}\right) + \frac{1}{2}\left(e^{j2\pi 104t} + e^{-j2\pi 104t}\right)$$

At 0Hz, there is no energy, so the complete spectrum has these frequencies and amplitudes:

$$\begin{aligned}(f_{-2}, a_{-2}) &= (-110, 0.5)\\ (f_{-1}, a_{-2}) &= (-104, 0.5)\\ (f_0, a_0) &= (0, 0)\\ (f_1, a_1) &= (104, 0.5)\\ (f_2, a_2) &= (110, 0.5)\end{aligned}$$

One could analyze these as the harmonics of a 2Hz fundamental, in which case, for $T_0 = 0.5$ seconds, we would have

$$X_k = \begin{cases} 0.5 & k \in \{-55, -52, 52, 55\} \\ 0 & \text{otherwise} \end{cases}$$

Problem 2.4

Suppose that a violin is playing the note A4 (440Hz), but our recording quality is bad, so we only get the first two harmonics:

$$x(t) = \sum_{k=-2}^{2} a_k e^{j2\pi k 440 t}$$

Suppose we measure the spectrum, and find it to be

$$\{(-880, 0.01), (-440, 1), (0, 0), (440, 1), (880, 0.01)\}$$

In order to improve the balance a little, we try differentiating the tone. Our differentiator also imposes a delay and a DC offset, though, so what we get is

$$y(t) = \frac{dx(t - 0.001)}{dt} + 1.5$$

Find the spectrum of $y(t)$.

Solution: The time delay multiplies each coefficient by a phase offset term, $e^{-j2\pi f_k \tau}$. Differentiation multiplies each coefficient by $j2\pi f_k$. The DC offset just adds 1.5 to the zeroth spectral coefficient. The resulting spectrum is

$$\left\{\left(-880, -j2\pi 8.8e^{j2\pi 0.88}\right), \left(-440, -j2\pi 440e^{j2\pi 0.44}\right), (0, 1.5), \left(440, j2\pi 440e^{-j2\pi 0.44}\right), \left(880, j2\pi 8.8e^{-j2\pi 0.88}\right)\right\}$$

which can be written as

$$\left\{\left(-880, 2\pi 8.8e^{j(2\pi 0.88-\frac{\pi}{2})}\right), \left(-440, 2\pi 440e^{j(2\pi 0.44-\frac{\pi}{2})}\right), (0, 1.5), \left(440, 2\pi 440e^{-j(2\pi 0.44-\frac{\pi}{2})}\right), \left(880, 2\pi 8.8e^{-j(2\pi 0.88-\frac{\pi}{2})}\right)\right\}$$

or equivalently

$$\left\{\left(-880, 17.6\pi e^{j1.26\pi}\right), \left(-440, 880\pi e^{j0.38\pi 0.44}\right), (0, 1.5), \left(440, 880\pi e^{-j0.38\pi}\right), \left(880, 17.6\pi e^{-j1.26\pi}\right)\right\}$$