ECE 401 Signal and Image Analysis
Homework 5

UNIVERSITY OF ILLINOIS
Department of Electrical and Computer Engineering

Assigned: Monday, 11/2/2020; Due: Monday, 11/9/2020
Reading: *DSP First* Sections 10.11-10.12

Problem 5.1

What is $h[n]$ if

$$H(z) = \frac{1}{(1-e^{j0.1\pi}z^{-1})(1-e^{-j0.1\pi}z^{-1})}$$

Solution: Using PFE, we get

$$H(z) = \frac{C_1}{1-e^{j0.1\pi}z^{-1}} + \frac{C_1^*}{1-e^{-j0.1\pi}z^{-1}}$$

Solving for C_1, we can find that $C_1 = p_1/(p_1 - p_1^*) = e^{j0.1\pi}/(2j\sin(0.1\pi))$, so

$$h[n] = \left(\frac{e^{j0.1\pi(n+1)}}{2j\sin(0.1\pi)} - \frac{e^{-j0.1\pi(n+1)}}{2j\sin(0.1\pi)}\right)u[n] = \frac{\sin(0.1\pi(n+1))}{\sin(0.1\pi)}u[n]$$

Problem 5.2

Consider a second-order resonator with a resonant frequency of $F_1 = 500$Hz and a bandwidth of $B_1 = 400$Hz, sampled at $F_s = 16000$samples/second. What are $H(z)$ and $h[n]$?

Solution:

$$\omega_1 = \frac{2\pi F_1}{F_s} = \frac{2\pi 500}{16000} = \frac{\pi}{16}$$
$$\sigma = \frac{1}{2}\frac{2\pi B_1}{F_s} = \frac{\pi 400}{16000} = \frac{\pi}{40}$$
$$H(z) = \frac{1}{(1-e^{-\sigma+j\omega_1}z^{-1})(1-e^{-\sigma-j\omega_1}z^{-1})}$$
$$h[n] = \frac{1}{\sin(\omega_1)}e^{-\sigma n}\sin(\omega_1(n+1))u[n]$$

Problem 5.3

Suppose

$$x[n] = \frac{1}{\sin(0.3\pi)} e^{-0.1(n-6)} \sin\left(0.3\pi(n-5)\right) u[n-6]$$

Write a difference equation in terms of $y[n]$ and $x[n]$ that will result in $y[n] = \delta[n-6]$.

Solution: This $x[n]$ is the result of passing $y[n] = \delta[n-6]$ through a second-order resonator,

$$H(z) = \frac{1}{(1-p_1 z^{-1})(1-p_1^* z^{-1})}$$

where $p_1 = e^{-0.1+j0.3\pi}$. We can get $y[n]$ back again by passing $x[n]$ through the inverse filter,

$$A(z) = (1-p_1 z^{-1})(1-p_1^* z^{-1}) = 1 - 2e^{-0.1}\cos(0.3\pi)z^{-1} + e^{-0.2}z^{-2}$$

Implementing $A(z)$ as a difference equation, we get

$$y[n] = x[n] - 2e^{-0.1}\cos(0.3\pi)x[n-1] + e^{-0.2}x[n-2]$$

Problem 5.4

Suppose $x[n]$ is a signal with autocorrelation coefficients $R[0] = 1$, $R[1] = 0.5$, and $R[2] = 0.5$. Find coefficients a_1 and a_2 that will minimize $\mathcal{E}$, which is defined as

$$\mathcal{E} = \sum_{n=-\infty}^{\infty} \left(x[n] - a_1 x[n-1] - a_2 x[n-2]\right)^2$$

Solution: If we set $\frac{d\mathcal{E}}{da_1} = 0$ and $\frac{d\mathcal{E}}{da_2} = 0$, we get two equations in two unknowns, which can be written in matrix form as

$$\begin{bmatrix} R[1] \\ R[2] \end{bmatrix} = \begin{bmatrix} R[0] & R[1] \\ R[1] & R[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

We solve by inverting the matrix, which gives

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \frac{1}{1-(0.5)(0.5)} \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$$