UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 417 PRINCIPLES OF SIGNAL ANALYSIS
Spring 2015

EXAM 1 SOLUTIONS

Thursday, February 19, 2015

Problem 1 (25 points)

A particular speech signal has the following log magnitude DTFT:

$$\log |X(\omega)| = \frac{2\pi G}{T_0} \sum_{k=0}^{T_0-1} \delta\left(\omega - \frac{2\pi k}{T_0}\right) + \sum_{m=1}^{4} h_m \cos(m\omega)$$

for some constants G, T_0, and h_m; assume that $T_0 > 4$. Define

$$\begin{aligned} \hat{x}[n] &= \text{DTFT}^{-1}\left(\log|X(\omega)|\right) \\ \hat{g}[n] &= w[n]\hat{x}[n] \\ \log G_M(\omega) &= \text{DTFT}\left(\hat{g}[n]\right) \end{aligned}$$

and

$$w[n] = \begin{cases} 1 & |n| \le M \\ 0 & \text{otherwise} \end{cases}$$

Specify the value of $\log G_M(\omega)$, as a simple function of both M and ω, in terms of G, T_0, and h_m, for **every possible positive value of M**

SOLUTION:

$$\log G_M(\omega) = \begin{cases} G + \sum_{m=1}^{M} h_m \cos(m\omega) & m \le 4 \\ G + \sum_{m=1}^{4} h_m \cos(m\omega) + \sum_{k=1}^{\ell} 2G\cos(kT_0\omega) & \ell T_0 \le M < (\ell+1)T_0 \end{cases}$$

Problem 2 (20 points)

A particular dataset has three data,

$$\vec{x}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Define $X = [\vec{x}_1, \vec{x}_2, \vec{x}_3]$ and $R = X^T X$. The matrix R is given by $R = V\Lambda V^T$ where

$$V = \begin{bmatrix} -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

Find a matrix W such that $\vec{y}_i = W^T\vec{x}_i$, $\vec{y}_i$ is two-dimensional, and the elements of $\vec{y}_i$ are uncorrelated.

SOLUTION:

$$W = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \\ \sqrt{\frac{3}{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 \\ -\sqrt{\frac{3}{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

...or any matrix whose columns are proportional to the columns shown above (because the problem asked for uncorrelated elements, but did not specify the variance of each element).

Problem 3 (15 points)

Let $a_i[n_1, n_2]$ denote the $(n_1, n_2)^{\text{th}}$ pixel of a grayscale image a_i, where $0 \le n_1 \le N_1 - 1$ and $0 \le n_2 \le N_2 - 1$, therefore the vectorized version of the same image, $\vec{x}_i$, is an (N_1N_2)-dimensional vector. For this problem it does not matter whether you vectorize the image in row order or in column order.

Suppose that the first image in the training database is a fuzzy striped image, given by

$$a_1[n_1, n_2] = \frac{255}{2} + \frac{255}{2}\cos\left(\frac{2\pi n_2}{6}\right)$$

Suppose that pixel values are constrained to be non-negative, and to lie in the range $0 \le a_i[n_1, n_2] \le 255$.

Under these constraints, find the image $a_2[n_1, n_2]$ that maximizes $\|\vec{x}_2\|^2$ subject to the constraint $\vec{x}_1^T\vec{x}_2 = 0$. **Hint:** First figure out which pixels can be nonzero, then figure out what their values must be.

SOLUTION:

$$a_2[n_1, n_2] = \begin{cases} 255 & n_2 = 3 + 6k \text{ for any integer } k \\ 0 & \text{otherwise} \end{cases}$$

Problem 4 (15 points)

Suppose you have a 16000-sample audio waveform, $x[n]$, such that $x[n] \ne 0$ for $0 \le n \le 15999$. You want to chop this waveform into 400-sample frames, with 10% overlap between frames. How many nonzero samples are there in the last frame?

SOLUTION: Nonzero samples $= L + (N-1)S - 16000 = 160$ samples, where $L = 400$ is the frame length, $S = 360$ is the frame skip, and $N = 45$ is the number of frames.

Problem 5 (25 points)

Suppose you have a database with three samples from class 0, and two samples from class 1, in other words, the data labels are

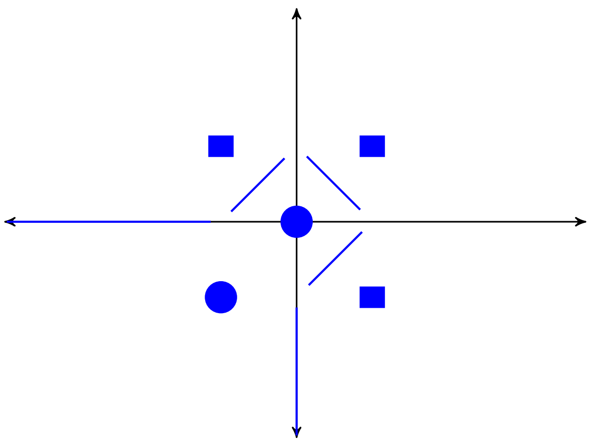
$$Y = [y_1, y_2, y_3, y_4, y_5] = [0, 0, 0, 1, 1]$$

Each observation is an $\Re^2$-vector, and they are given by

$$X = [\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4, \vec{x}_5] = \begin{bmatrix} -1 & 1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 & -1 \end{bmatrix}$$

(a) Consider a nearest-neighbor (NN) classifier. In a two-dimensional vector space, show the boundary that separates class 0 from class 1. Label the coordinates of every discontinuity.

SOLUTION: Discontinuities at $(-1, 0)$, $(0, 1)$, $(1, 0)$, and $(0, -1)$.



(b) Repeat part (a), but this time for a 3NN (3-nearest-neighbor) classifier.

SOLUTION: The line $x_2 = -x_1$.