UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 417 Multimedia Signal Processing
Fall 2018

EXAM 1

Thursday, October 18, 2018

- This is a CLOSED BOOK exam. You may use one sheet (front and back) of hand-written notes.
- No calculators are permitted. You need not simplify explicit numerical expressions.
- There are a total of 100 points in the exam. Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

Problem	Score
1	
2	
3	
4	
5	
6	
Total	

Name: ______________________________

Possibly Useful Formulas

Fourier Transforms

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega \leftrightarrow X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N} \leftrightarrow X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

$$\delta[n] \leftrightarrow 1, \qquad e^{j\alpha n} \leftrightarrow 2\pi\delta(\omega-\alpha)$$

$$\sum_{\ell=-\infty}^{\infty} \delta[n-\ell T_0] \leftrightarrow \left(\frac{2\pi}{T_0}\right)\sum_{m=0}^{T_0-1}\delta\left(\omega - \frac{2\pi m}{T_0}\right)$$

$$h[n] * x[n] \leftrightarrow H(e^{j\omega})X(e^{j\omega}), \qquad w[n]x[n] \leftrightarrow \frac{1}{2\pi}W(e^{j\omega}) \circledast X(e^{j\omega})$$

Statistical, Signal, and Circular Autocorrelation Functions

$$R_{xx}[n] = E\left\{x[m]x[m-n]\right\}, \quad r_{xx}[n] = \sum_{m=-\infty}^{\infty} x[m]x[m-n], \quad \tilde{r}_{xx}[n] = \sum_{m=0}^{N-1} x[m]x\left[\langle m-n\rangle_N\right]$$

Gaussians, Mahalanobis, and PCA

$$\mathcal{N}\left(\vec{x};\vec{\mu},\Sigma\right) = \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}}e^{-\frac{1}{2}d_\Sigma^2(\vec{x},\vec{\mu})}$$

$$\Sigma = V\Lambda V^T, \quad V^TV = VV^T = I, \quad |\Sigma| = |\Lambda|$$

$$d_\Sigma^2(\vec{x},\vec{\mu}) = (\vec{x}-\vec{\mu})^T\Sigma^{-1}(\vec{x}-\vec{\mu}) = \vec{y}^T\Lambda^{-1}\vec{y}, \quad \vec{y} = V^T(\vec{x}-\vec{\mu})$$

Miscellaneous

$$\text{mel}(f) = 2595\log_{10}\left(1+\frac{f}{700}\right)$$

$$\cos(a)\cos(b) = \frac{1}{2}\cos(a-b) + \frac{1}{2}\cos(a+b)$$

Forward-Backward Algorithm

$$\alpha_1(i) = \pi_i b_i(\vec{x}_1), \quad \alpha_t(i) = \sum_{j=1}^{N}\alpha_{t-1}(j)a_{ji}b_i(\vec{x}_t)$$

$$\beta_T(i) = 1, \quad \beta_t(i) = \sum_{j=1}^{N}\beta_{t+1}(j)a_{ij}b_j(\vec{x}_{t+1})$$

Problem 1 (16 points)

A particular random signal $u[n]$ has the following DTFT:

$$U(e^{j\omega}) = ae^{j\theta}\delta(\omega - 0.2\pi) + ae^{-j\theta}\delta(\omega + 02.\pi)$$

where

- a is a real-valued Gaussian random variable with mean 0 and variance σ^2
- θ is a real-valued random variable uniformly distributed between 0 and 2π.

Find the random signal $u[n]$, and its statistical autocorrelation $R_{uu}[m]$, in terms of a, σ^2, θ, n, and/or m.

Problem 2 (17 points)

A particular voiced speech signal has pitch period P, and vocal tract transfer function $H(e^{j\omega})$. The signal is windowed by a window function $w[n]$ of length N, producing the windowed signal

$$s[n] = \begin{cases} w[n]\sum_{\ell=-\infty}^{\infty} h[n-\ell P] & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

Find $S[k]$, the N-point DFT of $s[n]$, in terms of k, P, N, $H(e^{j\omega})$, and $W(e^{j\omega})$.

Problem 3 (17 points)

Your goal is to find a positive real number, a, so that $ax[n]$ is as similar as possible to $y[n]$ in the sense that it minimizes the following error:

$$\epsilon = \int_{-\pi}^{\pi} \left(|Y(e^{j\omega})| - a|X(e^{j\omega})|\right)^2 d\omega$$

Find the value of a that mininimizes ϵ, in terms of $|X(e^{j\omega})|$ and $|Y(e^{j\omega})|$.

Problem 4 (17 points)

A 2-dimensional Gaussian random vector has mean $\vec{\mu}$ and covariance Σ given by

$$\vec{\mu} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

Draw a curve of some kind, on a two-dimensional Cartesian plane, showing the set of points $\left\{\vec{x} : p_X(\vec{x}) = \frac{1}{8\pi} e^{-\frac{1}{2}}\right\}$.

Problem 5 (16 points)

In terms of $\alpha_t(i)$, $\beta_t(i)$, a_{ij}, π_i and $b_i(\vec{x}_t)$, find

$$p\left(q_6 = i, q_7 = j | \vec{x}_1, \ldots, \vec{x}_{20}\right)$$

Problem 6 (17 points)

A particular HMM-based speech recognizer only knows two words: word w_0, and word w_1. Word w_0 has a higher *a priori* probability: $p_Y(w_0) = 0.7$, while $p_Y(w_1) = 0.3$. Each of the two words is modeled by a four-state Gaussian HMM ($N = 4$) with three-dimensional observations ($D = 3$). All states, in both HMMs, have identity covariance ($\Sigma_i = I$). Both HMMs have *exactly* the same transition probabilities and state-dependent means, given by:

$$\textbf{Both Words:}\ A = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix},\ \vec{\mu}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ \vec{\mu}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ \vec{\mu}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix},\ \vec{\mu}_4 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

But the initial residence probabilities are different:

$$\textbf{Word 0:}\ \pi_i = \begin{cases} 1 & i = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \textbf{Word 1:}\ \pi_i = \begin{cases} 1 & i = 4 \\ 0 & \text{otherwise} \end{cases}$$

Suppose that you have a two-frame observation, $X = [\vec{x}_1, \vec{x}_2]$, where $\vec{x}_t = [x_{1t}, x_{2t}, x_{3t}^T]$. The MAP decision rule, in this case, can be written as a linear classifier,

$$\hat{y} = \begin{cases} w_1 & \vec{w}_1^T \vec{x}_1 + \vec{w}_2^T \vec{x}_2 + b > 0 \\ w_0 & \text{otherwise} \end{cases}$$

Find $\vec{w}_1$, $\vec{w}_2$, and b.