UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 417 Multimedia Signal Processing
Fall 2019

EXAM 1

Tuesday, September 24, 2018

- This is a CLOSED BOOK exam. You may use one sheet (front and back) of hand-written notes.
- No calculators are permitted. You need not simplify explicit numerical expressions.
- There are a total of 50 points in the exam. Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

Problem	Score
1	
2	
3	
4	
5	
Total	

Name: ____

Possibly Useful Formulas

Fourier Transforms

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \leftrightarrow x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}} \leftrightarrow x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[j]e^{j\frac{2\pi kn}{N}}$$

$$x[n] = e^{j\omega_0 n} \leftrightarrow X(e^{j\omega}) = 2\pi\delta(\omega - \omega_0)$$

$$w_R[n] = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases} \leftrightarrow W_R(\omega) = \frac{\sin(\omega N/2)}{\sin(\omega/2)}e^{-j\omega(\frac{N-1}{2})}$$

Autocorrelation and Power Spectrum

$$R_{xx}[n] = E\left\{x[m]x[m-n]\right\} \leftrightarrow S_{xx}(\omega) = \sum_{n=-\infty}^{\infty} R_{xx}[n]e^{-j\omega n}$$

$$r_{xx}[n] = \sum_{m=-\infty}^{\infty} x[m]x[m-n] \leftrightarrow s_{xx}(\omega) = \sum_{n=-\infty}^{\infty} r_{xx}[n]e^{-j\omega n}$$

Gaussians, Mahalanobis, and PCA

$$\mathcal{N}\left(\vec{x}; \vec{\mu}, \Sigma\right) = \frac{1}{(2\pi)^{D/2}|R|^{1/2}}e^{-\frac{1}{2}d_R^2(\vec{x},\vec{\mu})}$$

$$R = V\Lambda V^T, \quad V^T V = VV^T = I, \quad |R| = |\Lambda|$$

$$d_R^2(\vec{x},\vec{\mu}) = (\vec{x}-\vec{\mu})^T R^{-1}(\vec{x}-\vec{\mu}) = \vec{y}^T\Lambda^{-1}\vec{y}, \quad \vec{y} = V^T(\vec{x}-\vec{\mu})$$

Problem 1 (20 points)

A particular signal, $x[n]$, is sampled at $F_s = 18,000$ samples/second. There are a total of $10,000$ samples, numbered $x[0]$ through $x[9999]$. These samples are divided into T frames, $\vec{x}_t$, with a framelength of 250 samples and a frame skip of 100 samples, i.e.,

$$\vec{x}_t = \begin{bmatrix} x[100t] \\ \vdots \\ x[100t+249] \end{bmatrix}$$

Your goal is to create two different $480 \times T$ matrices: $X = [\vec{X}_0, \ldots, \vec{X}_{T-1}]$ is the STFT (short-time Fourier transform) of $x[n]$, and $S = [\vec{S}_0, \ldots, \vec{S}_{T-1}]$ is the spectrogram of $x[n]$. The final image matrix S should show the spectral level (in decibels) of $x[n]$, as a function of time and frequency, in the frequency range from 0Hz to 5000Hz.

(a) Find T, the number of frames. This should be set so that (1) every sample of $x[n]$ appears in at least one frame, and (2) there is at most one frame with zero-padding. Your answer should be a number, or an explicit numerical expression.

(b) Each STFT vector, $\vec{X}_t$, is the length-N DFT of one frame $\vec{x}_t$. Find N. Your answer should be a number, or an explicit numerical expression.

(c) The STFT is given by $\vec{X}_t = A\vec{x}_t$ for some matrix, A, whose $(k,n)^{\text{th}}$ element is a_{kn}. Give an expression for a_{kn} in terms of k, n, and N.

(d) Suppose that $X_{max} = \max_k \max_t |X[k,t]|$. The spectrogram $S[k,t]$ is the level of $X[k,t]$, in decibels, scaled so that $0 \leq S[k,t] \leq 255$, and so that $S[k,t] = 0$ if and only if $|X[k,t]| \leq X_{max}/1000$. Give an equation specifying $S[k,t]$ as a function of $X[k,t]$.

Problem 2 (5 points)

The signal $x[n]$ is given by

$$x[n] = \begin{cases} \cos(\omega_0 n) & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

$X[k]$ is the length-N DFT of $x[n]$. Find $X[k]$, in terms of N and ω_0. You may find it useful to write your answer in terms of the transform of a rectangular window, $W_R(\omega)$, which is

$$w_R[n] = \begin{cases} 1 & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases} \leftrightarrow W_R(\omega) = \frac{\sin(\omega N/2)}{\sin(\omega/2)} e^{-j\omega\left(\frac{N-1}{2}\right)}$$

Problem 3 (5 points)

A random signal $x[n]$ has the autocorrelation function $R_{xx}[n] = \rho^{|n|}$, for some real constant $0 < \rho < 1$. Find its power spectrum $S_{xx}(\omega)$, in terms of ω and ρ. Your answer should contain no infinite-length summations.

Problem 4 (5 points)

$x[n]$ is a signal with N samples, numbered $x[0]$ through $x[N-1]$. Find M and $s[m]$ so that

(a) every sample of $s[m]$ is either $s[m] = 0$ or $s[m] = x[n]$ for some n,

(b) every sample of $x[n]$ is used at least once,

(c) $S[k]$, the M-point DFT of $s[n]$, is real-valued.

You may define the sample times m to be non-integers, if you wish, though correct answers with integer-valued sample times also exist.

Problem 5 (15 points)

Suppose you have an $M \times D$ matrix, $X = [\vec{x}_0, \ldots, \vec{x}_{M-1}]^T$, where $\sum_{m=0}^{M-1} \vec{x}_m = \vec{0}$. The eigenvalues of $X^T X$ are λ_0 through λ_{D-1}, its eigenvectors are $\vec{v}_0$ through $\vec{v}_{D-1}$, and its principal components are $Y = XV$.

(a) Write $Y^T Y$ in terms of the eigenvalues, λ_0 through λ_{D-1}.

(b) Write $\sum_{m=0}^{M-1} \|\vec{x}_m\|_2^2$ in terms of the eigenvalues, λ_0 through λ_{D-1}.

(c) Write $\vec{v}_i^T X^T X \vec{v}_j$ in terms of the eigenvalues, λ_0 through λ_{D-1}, for $0 \leq i \leq j \leq D-1$.