UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 417 Multimedia Signal Processing
Fall 2019

EXAM 3

Tuesday, December 17, 2019, 1:30-4:30pm

Problem 1 (15 points)

The Maesters of the Citadel need to determine when winter starts. The temperature on day t is x_t. The state of day t is either $q_t = 0$ (Autumn) or $q_t = 1$ (Winter). Nobody really knows how cold this winter will be or how long it will last, but the Maesters have created an initial model $\Lambda = \{a_{ij}, b_j(x)\}$ where $a_{ij} \equiv p(q_t = j|q_{t-1} = i)$ and $b_j(x) \equiv p(x_t = x|q_t = j)$.

(a) Suppose we have a particular three day sequence of measurements, x_1, x_2, and x_3. Given that the preceding day was still autumn ($q_0 = 0$), we want to determine the joint probability that it continued to be autumn for days 1, 2, and 3, and that the three observed temperatures were measured. In other words, we want an estimate of

$$G_1 = p(q_1 = 0, x_1, q_2 = 0, x_2, q_3 = 0, x_3|q_0 = 0, \Lambda)$$

Find G_1 in terms of a_{ij} and $b_j(x_t)$, for whatever particular values of i, j, and t are most useful to you.
Solution:

$$G_1 = a_{00}^3 b_0(x_1)b_0(x_2)b_0(x_3)$$

(b) Suppose it is known that the preceding day was still autumn ($q_0 = 0$). Now, on day 1, the Maesters have determined that the temperature is x_1. Find the conditional probability, given this measurement, that it is still autumn, i.e., find

$$G_2 = p(q_1 = 0|x_1, q_0 = 0, \Lambda)$$

Find G_2 in terms of a_{ij} and $b_j(x_t)$, for whatever particular values of i, j, and t are most useful to you.
Solution:

$$G_2 = \frac{p(q_1 = 0, x_1|q_0 = 0, \Lambda)}{\sum_i p(q_1 = i, x_1|q_0 = 0, \Lambda)} = \frac{a_{00}b_0(x_1)}{a_{00}b_0(x_1) + a_{01}b_1(x_1)}$$

(c) The Maesters have collected a long series of measurements, $\{x_1, \ldots, x_T\}$ for T consecutive days. From these measurements, the Maesters have applied the forward-backward algorithm in order to calculate the following two quantities:

$$\alpha_t(i) \equiv p(x_1, \ldots, x_t, q_t = i|\Lambda), \quad \beta_t(i) \equiv p(x_{t+1}, \ldots, x_T|q_t = i, \Lambda)$$

Using these quantities, the Maesters wish to calculate the probability that Winter started on a particular day, $t = w$. That is, they wish to find

$$G_3 = p(q_{w-1} = 0, q_w = 1 | x_1, \dots, x_T, \Lambda)$$

Find G_3 in terms of $\alpha_t(i)$, $\beta_t(i)$, a_{ij} and $b_j(x_t)$, for whatever particular values of i, j, and t are most useful to you.

Solution:

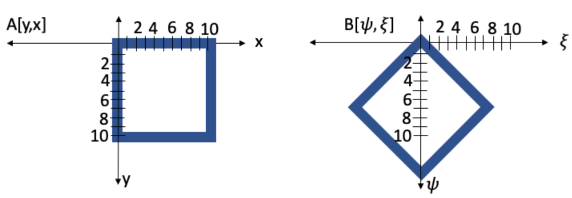
$$G_3 = \frac{p(q_{w-1} = 0, q_w = 1, x_1, \dots, x_T | \Lambda)}{\sum_i \sum_j p(q_{w-1} = i, q_w = j, x_1, \dots, x_T | \Lambda)} = \frac{\alpha_{w-1}(0) a_{01} b_1(x_w) \beta_w(1)}{\sum_i \sum_j \alpha_{w-1}(i) a_{ij} b_j(x_w) \beta_w(j)}$$

Problem 2 (15 points)

Suppose you have a picture of a white square on a black field, $A[y, x]$, where x is the column index, y is the row index. You wish to perform an affine transform that will turn your square into a picture of a diamond, $B[\psi, \xi]$, in which ξ is the column index, and ψ is the row index:

$$A[y, x] = \begin{cases} 255 & x = 0 \text{ or } x = 10, \quad 0 \le y \le 10 \\ 255 & y = 0 \text{ or } y = 10, \quad 0 \le x \le 10 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$B[\psi, \xi] = \begin{cases} 255 & \psi - \xi = 0 \text{ or } \psi - \xi = 10\sqrt{2}, \quad 0 \le \psi + \xi \le 10\sqrt{2} \\ 255 & \psi + \xi = 0 \text{ or } \psi + \xi = 10\sqrt{2}, \quad 0 \le \psi - \xi \le 10\sqrt{2} \\ 0 & \text{otherwise} \end{cases}$$



(a) **Affine Transform:** This affine transform can be written by a transform matrix, as

$$\begin{bmatrix} \xi \\ \psi \\ 1 \end{bmatrix} = \begin{bmatrix} a, b, c \\ d, e, f \\ g, h, i \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Find a, b, c, d, e, f, g, h and i. Your answers should be numbers, or else explicit numerical expressions; there should be no unresolved variables in your answers. Note: there is more than one correct answer.

Solution:

There are exactly eight correct solutions. The two solutions that map point $x = 0, y = 0$ to point $\xi = 0, \psi = 0$ are given by

$$\begin{bmatrix} a, b, c \\ d, e, f \\ g, h, i \end{bmatrix} = \begin{bmatrix} \pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}, 0 \\ \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \\ 0, 0, 1 \end{bmatrix}$$

The two solutions that map point $x = 10, y = 0$ to point $\xi = 0, \psi = 0$ are given by

$$\begin{bmatrix} a, b, c \\ d, e, f \\ g, h, i \end{bmatrix} = \begin{bmatrix} \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}, \mp 5\sqrt{2} \\ -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 5\sqrt{2} \\ 0, 0, 1 \end{bmatrix}$$

The two solutions that map point $x = 10, y = 10$ to point $\xi = 0, \psi = 0$ are given by

$$\begin{bmatrix} a, b, c \\ d, e, f \\ g, h, i \end{bmatrix} = \begin{bmatrix} \mp\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}, 0 \\ -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 10\sqrt{2} \\ 0, 0, 1 \end{bmatrix}$$

The two solutions that map point $x = 0, y = 10$ to point $\xi = 0, \psi = 0$ are given by

$$\begin{bmatrix} a, b, c \\ d, e, f \\ g, h, i \end{bmatrix} = \begin{bmatrix} \mp\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}, \pm 5\sqrt{2} \\ \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 5\sqrt{2} \\ 0, 0, 1 \end{bmatrix}$$

(b) **Bilinear Interpolation:** $A[y, x]$ is a discrete-space image (y and x are integers), whereas $A(y, x)$ is the corresponding continuous-space image (y and x are real numbers). An affine transform maps integer coordinates ξ and ψ to real-valued coordinates x and y, so it's necessary to estimate the values of the continuous-space image. For example, one way to compute the intensity of the pixel at $B[2, 1]$ is by setting it equal to $A\left(\frac{3\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \approx A(2.1, 0.7)$. Use bilinear interpolation, combined with the information given in Eq. 1, to estimate the numerical value of $A(2.1, 0.7)$.

Solution:

$$\begin{aligned} A(2.1, 0.7) &= (0.9)(0.3)A[2, 0] + (0.9)(0.7)A[2, 1] + (0.1)(0.3)A[3, 0] + (0.1)(0.7)A[3, 1] \\ &= (0.9)(0.3)(255) + (0.9)(0.7)(0) + (0.1)(0.3)(255) + (0.1)(0.7)(0) \\ &= (0.3)255 \end{aligned}$$

(The last step, simplification, is optional).

(c) **Barycentric Coordinates:** Suppose we have some coordinate with known values of x and y, and we're trying to find the values of ξ and ψ to which it gets moved. One way to solve this problem is by using the transform matrix you computed in part (a). A different solution is to use the triangle whose coordinates are $[x_1, y_1]$, $[x_2, y_2]$, and $[x_3, y_3]$ before transformation, but $[\xi_1, \psi_1]$, $[\xi_2, \psi_2]$, and $[\xi_3, \psi_3]$ after transformation, where

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} \xi_1 \\ \psi_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \xi_2 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} 5\sqrt{2} \\ 5\sqrt{2} \end{bmatrix}, \quad \begin{bmatrix} \xi_3 \\ \psi_3 \end{bmatrix} = \begin{bmatrix} -5\sqrt{2} \\ 5\sqrt{2} \end{bmatrix}$$

In terms of these triangles, the Barycentric coordinates of any point are the same before and after transformation:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \lambda_1 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \lambda_2 \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \lambda_3 \begin{bmatrix} x_3 \\ y_3 \end{bmatrix}, \quad \begin{bmatrix} \xi \\ \psi \end{bmatrix} = \lambda_1 \begin{bmatrix} \xi_1 \\ \psi_1 \end{bmatrix} + \lambda_2 \begin{bmatrix} \xi_2 \\ \psi_2 \end{bmatrix} + \lambda_3 \begin{bmatrix} \xi_3 \\ \psi_3 \end{bmatrix}$$

where $\lambda_1 + \lambda_2 + \lambda_3 = 1$. Suppose that x and y are known, but ξ and ψ are unknown. **Find λ_1, λ_2, and λ_3 in terms of x and y.**
Solution:

$$\lambda_2 = \frac{x}{10}, \quad \lambda_3 = \frac{y}{10}, \quad \lambda_1 = 1 - \lambda_2 - \lambda_3$$

Problem 3 (20 points)

Suppose we're trying to predict the sequence $\zeta_1, \ldots, \zeta_{100}$ from the sequence $x_1, \ldots, x_{100}$. We want to use some type of neural net (fully-connected, CNN, or LSTM) to compute $z_1, \ldots, z_{100}$ in order to minimize the error

$$E = \frac{1}{200} \sum_{t=1}^{100} (z_t - \zeta_t)^2$$

We only have one training sequence $(x_1, \ldots, x_{100}, \zeta_1, \ldots, \zeta_{100})$.

(a) Suppose we use a **fully-connected one-layer neural net**, with 10,000 trainable network weights w_{kj}, and 100 trainable bias terms w_{k0}, such that

$$z_k = \sigma\left(w_{k0} + \sum_{j=1}^{100} w_{kj} x_j\right)$$

where $\sigma(x) = 1/(1 + e^{-x})$ is the logistic nonlinearity. Find the derivatives of the error with respect to the weights (dE/dw_{kj}) and biases (dE/dw_{k0}). Express your answers in terms of x_j, z_k, and ζ_k for appropriate values of k and j; **the terms w_{kj} and w_{k0} should not show up on the right-hand-side of any of your equations**.
Solution:

$$\frac{dE}{dw_{k0}} = \frac{1}{100}(z_k - \zeta_k) z_k (1 - z_k)$$
$$\frac{dE}{dw_{kj}} = \frac{1}{100}(z_k - \zeta_k) z_k (1 - z_k) x_j$$

(b) Suppose we use a **CNN (convolutional neural net)** with 99 trainable weights $w[\tau]$ and a single scalar bias term, b, i.e.,

$$z_t = \sigma\left(b + \sum_{\tau=-49}^{49} w[\tau] x_{t-\tau}\right)$$

where $\sigma(x) = 1/(1 + e^{-x})$ is the logistic nonlinearity. Find the derivatives of the error with respect to the weights ($dE/dw[\tau]$) and bias (dE/db). Assume that $x_t = 0$ for $t \leq 0$ or $t \geq 101$. Express your answers in terms of x_j, z_k, and ζ_k for appropriate values of k and j; **the terms $w[\tau]$ and b should not show up on the right-hand-side of any**

of your equations.

Solution:

$$\frac{dE}{db} = \frac{1}{100}\sum_{t=1}^{100}(z_t - \zeta_t)z_t(1-z_t)$$

$$\frac{dE}{dw[\tau]} = \frac{1}{100}\sum_{t=1}^{100}(z_t - \zeta_t)z_t(1-z_t)x_{t-\tau}$$

(c) Suppose we use an **RNN (recurrent neural network)** with just one scalar memory cell whose weights and biases are w, u, and b:

$$z_t = \sigma(ux_t + wz_{t-1} + b)$$

Find the derivatives of the error with respect to the weights and biases (dE/du, dE/dw, and dE/db). Express your answers in terms of x_j, z_k, and ζ_k for appropriate values of k and j; **the terms u, w and b should not show up on the right-hand-side of any of your equations.** You may express your answer recursively, or your answer may contain summation ($\sum$) and/or product ($\prod$) terms.

Solution:

It is possible to express dE/du, dE/dw, and dE/db without w on the right-hand-side only if we express them in terms of dE/dz_t. It is not possible to correctly express dE/dz_t without w on the right-hand-side. This should probably be considered to be an error in the problem statement, but in any case, here's the solution I find that is closest to matching the problem statement:

$$\frac{\partial E}{\partial z_t} = \frac{1}{100}(z_t - \zeta_t)$$

$$\begin{aligned}\frac{dE}{dz_t} &= \frac{\partial E}{\partial z_t} + \frac{dE}{dz_{t+1}}\frac{\partial z_{t+1}}{\partial z_t} \\ &= \frac{1}{100}(z_t - \zeta_t) + \frac{dE}{dz_{t+1}}z_{t+1}(1-z_{t+1})w\end{aligned}$$

$$\frac{dE}{db} = \sum_{t=1}^{100} \frac{dE}{dz_t}\frac{\partial z_t}{\partial b}$$
$$= \sum_{t=1}^{100} \frac{dE}{dz_t} z_t(1-z_t)$$
$$\frac{dE}{du} = \sum_{t=1}^{100} \frac{dE}{dz_t}\frac{\partial z_t}{\partial u}$$
$$= \sum_{t=1}^{100} \frac{dE}{dz_t} z_t(1-z_t)x_t$$
$$\frac{dE}{dw} = \sum_{t=1}^{100} \frac{dE}{dz_t}\frac{\partial z_t}{\partial w}$$
$$= \sum_{t=2}^{100} \frac{dE}{dz_t} z_t(1-z_t)z_{t-1}$$

(d) Suppose we use an **LSTM (long-short-term memory network)** whose weights and biases are pre-specified: $u_c = 1$, and all of the other weights and biases are zero:

$$b_c = 0, u_c = 1, w_c = 0, b_f = 0, u_f = 0, w_f = 0, b_i = 0, u_i = 0, w_i = 0, b_o = 0, u_o = 0, w_o = 0$$

$$f[t] = \sigma(u_f x_t + w_f z_{t-1} + b_f), \quad i[t] = \sigma(u_i x_t + w_i z_{t-1} + b_i), \quad o[t] = \sigma(u_o x_t + w_o z_{t-1} + b_o)$$
$$c[t] = f[t]c[t-1] + i[t]\sigma\left(u_c x_t + w_c z_{t-1} + b_c\right), \quad z_t = o[t]c[t]$$

Assume that $c[t] = 0$ for $t \leq 0$. **Express z_t in terms of $\sigma(x_t)$ for** $0 \leq t \leq 100$. **Your answer should NOT contain any of the variables $c[t]$, $f[t]$, $i[t]$, or $o[t]$. Your answer may contain a summation ($\sum$).** You may find it useful to know that $\sigma(0) = \frac{1}{2}$.

Solution:

$$c[t] = \frac{1}{2}c[t-1] + \frac{1}{2}\sigma(x_t)$$
$$z_t = \frac{1}{2}c[t]$$
$$= \sum_{\tau=1}^{t} \left(\frac{1}{2}\right)^{2+t-\tau} \sigma(x_\tau)$$