UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 417 Multimedia Signal Processing
Fall 2019

EXAM 2

Tuesday, October 22, 2019

Problem 1 (10 points)

Consider an unvoiced zero-mean Gaussian random signal, $x[n] \sim \mathcal{N}(0,1)$, with the following autocorrelation:

$$R_{xx}[n] = e^{-\beta|n|}$$

Suppose $e[n] = x[n] - \alpha x[n-1]$.

(a) Find α to minimize $E\left[e^2[n]\right]$.
Solution:

$$\begin{aligned}\varepsilon &= E\left[e^2[n[\right] = E\left[(x[n] - \alpha x[n-1])^2\right] \\ &= R_{xx}[0] - 2\alpha R_{xx}[1] + \alpha^2 R_{xx}[0] \\ \frac{\partial \varepsilon}{\partial \alpha} &= -2R_{xx}[1] + 2\alpha R_{xx}[0] \\ \alpha &= \frac{R_{xx}[1]}{R_{xx}[0]} = e^{-\beta}\end{aligned}$$

(b) Find the value of $E\left[e^2[n]\right]$ that results from the α you chose in part (a).
Solution:

$$\begin{aligned}\varepsilon &= R_{xx}[0] - 2\alpha R_{xx}[1] + \alpha^2 R_{xx}[0] \\ &= 1 - 2\alpha e^{-\beta} + \alpha^2 \\ &= 1 - e^{-2\beta}\end{aligned}$$

Problem 2 (5 points)

Consider the synthesis filter $s[n] = e[n] + bs[n-1] - \left(\frac{b}{2}\right)^2 s[n-2]$. For what values of b is the synthesis filter stable?
Solution:

Take the Z transform of the difference equation and re-arrange terms, we get

$$S(z)(1 - bz^{-1} + \left(\frac{b}{2}\right)^2 z^{-2}) = E(z)$$

is stable if the roots of the polynomial have absolute value less than 1. The roots of the polynomial are

$$r_k = \frac{b}{2} \pm \frac{\sqrt{b^2 - 4(b/2)^2}}{2} = \frac{b}{2}$$

So $|r_k| < 1$ iff $|b| < 2$.

Problem 3 (5 points)

Suppose $e[n] \sim \mathcal{N}(0, 1)$ is Gaussian white noise, $R_{ee}[n] = \delta[n]$. Consider the synthesis filter, $s[n] = e[n] + \alpha s[n-1]$. Find the power spectrum of the synthesized signal, $S_{ss}(\omega)$, in terms of ω and α. You need not simplify, but your answer should contain no integrals or infinite sums.

Solution:

It was proven in class that $S_{ss}(\omega) = |H(e^{j\omega})|^2 S_{ee}(\omega)$. Since $S_{ee}(\omega) = \mathcal{F}\{R_{ee}[n]\} = 1$, we just need to find $H(e^{j\omega})$. To do that, we take the Z transform of the difference equation and re-arrange terms to get

$$H(z) = \frac{S(z)}{E(z)} = \frac{1}{1 - \alpha z^{-1}}$$

Subbing $z = e^{j\omega}$ gives

$$S_{ss}(\omega) = |H(e^{j\omega})|^2 = \left| \frac{1}{1 - \alpha e^{-j\omega}} \right|^2$$

Problem 4 (5 points)

You are given the integral image $ii[n_1, n_2]$, defined in terms of the image $i[n_1, n_2]$ as

$$ii[n_1, n_2] = \sum_{m_1=0}^{n_1} \sum_{m_2=0}^{n_2} i[m_1, m_2]$$

Write an equation that computes the complementary integral image, $c[n_1, n_2]$, in a small constant number of operations per output pixel, where

$$c[n_1, n_2] = \sum_{m_1=n_1}^{N_1-1} \sum_{m_2=n_2}^{N_2-1} i[m_1, m_2]$$

Solution:

$$\begin{aligned} c[n_1, n_2] &= \sum_{m_1=0}^{N_1-1} \sum_{m_2=0}^{N_2-1} i[m_1, m_2] - \sum_{m_1=0}^{n_1-1} \sum_{m_2=0}^{N_2-1} i[m_1, m_2] - \sum_{m_1=0}^{N_1-1} \sum_{m_2=0}^{n_2-1} i[m_1, m_2] + \sum_{m_1=0}^{n_1-1} \sum_{m_2=0}^{n_2-1} i[m_1, m_2] \\ &= ii[N_1 - 1, N_2 - 1] - ii[n_1 - 1, N_2 - 1] - ii[N_1 - 1, n_2 - 1] + ii[n_1 - 1, n_2 - 1] \end{aligned}$$

Problem 5 (5 points)

Suppose you have a 200×200-pixel image that is just one white dot at pixel $(45, 25)$, and all the other pixels are black:

$$x[n_1, n_2] = \begin{cases} 255 & n_1 = 45,\ n_2 = 25 \\ 0 & \text{otherwise},\ 0 \le n_1 < 199,\ 0 \le n_2 < 199 \end{cases}$$

This image is upsampled to size 400×400, then filtered, as

$$y[n_1, n_2] = \begin{cases} x[n_1/2, n_2/2] & n_1/2 \text{ and } n_2/2 \text{ both integers} \\ 0 & \text{otherwise} \end{cases} \qquad z[n_1, n_2] = y[n_1, n_2] * * h[n_1, n_2]$$

where $h[n_1, n_2]$ is the ideal anti-aliasing filter whose frequency response is

$$H(\omega_1, \omega_2) = \begin{cases} 1 & |\omega_1| < \frac{\pi}{2},\ |\omega_2| < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

Find $z[n_1, n_2]$.

Solution:

$$y[n_1, n_2] = \begin{cases} 255 & n_1 = 90, n_2 = 50 \\ 0 & \text{otherwise} \end{cases}$$

$$h[n_1, n_2] = \frac{1}{2}\text{sinc}\left(\frac{\pi n_1}{2}\right)\frac{1}{2}\text{sinc}\left(\frac{\pi n_2}{2}\right) = h_1[n_1]h_2[n_2]$$

Row convolution gives $v[n_1, n_2] = h_2[n_2] * y[n_1, n_2]$, which is

$$v[n_1, n_2] = \begin{cases} \frac{255}{2}\text{sinc}\left(\frac{\pi(n_2-50)}{2}\right) & n_1 = 90 \\ 0 & \text{otherwise} \end{cases}$$

Column convolution then gives

$$z[n_1, n_2] = \frac{255}{4}\text{sinc}\left(\frac{\pi(n_1 - 90)}{2}\right)\text{sinc}\left(\frac{\pi(n_2 - 50)}{2}\right)$$

Problem 6 (10 points)

Consider an infinite-sized RGB image containing a single diagonal white line on a black background, specifically

$$R[n_1, n_2] = G[n_1, n_2] = B[n_1, n_2] = \begin{cases} 255 & n_1 - n_2 = 5 \\ 0 & \text{otherwise} \end{cases}$$

where $-\infty < n_1 < \infty$, $-\infty < n_2 < \infty$, and the signals R, G, and B are the red, green, and blue channels, respectively.

(a) Find the luminance $Y[n_1, n_2]$.
Solution:

The best answer is

$$Y[n_1, n_2] = 255$$

because the coefficients of the first row of the transform matrix sum to 1. But an unsimplified explicit numerical formula is also OK:

$$Y[n_1, n_2] = 0.299 \times 255 + 0.587 \times 255 + 0.114 \times 255$$

(b) Find the blue-shift $P_b[n_1, n_2]$.
Solution:

The best answer is

$$P_b[n_1, n_2] = 0$$

because the coefficients of the second row of the transform matrix sum to 0. But an unsimplified explicit numerical formula is also OK:

$$P_b[n_1, n_2] = -0.168736 \times 255 - 0.331264 \times 255 + 0.5 \times 255$$

Problem 7 (10 points)

Consider an infinite-sized grayscale image of a diagonal gray line:

$$x[n_1, n_2] = \begin{cases} 105 & n_1 - n_2 = 5 \\ 0 & \text{otherwise} \end{cases}, \quad -\infty < n_1 < \infty, \quad -\infty < n_2 < \infty$$

(a) Suppose we **convolve each row** with a differencing filter:

$$y[n_1, n_2] = x[n_1, n_2] * d_2[n_2], \quad d_2[n_2] = \begin{cases} 1 & n_2 = 0 \\ -1 & n_2 = 2 \\ 0 & \text{otherwise} \end{cases}$$

Find $y[n_1, n_2]$.
Solution:

$$\begin{aligned} y[n_1, n_2] &= \sum_{m_2} x[n_1, n_2 - m_2] d_2[m_2] \\ &= x[n_1, n_2] - x[n_1, n_2 - 2] \\ &= \begin{cases} 105 & n_2 = n_1 - 5 \\ -105 & n_2 = n_1 - 3 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(b) Suppose, INSTEAD, that we **convolve each row** with an averaging filter

$$z[n_1,n_2] = x[n_1,n_2] * a_2[n_2], \quad a_2[n_2] = \begin{cases} 1 & n_2 \in \{0,2\} \\ 2 & n_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $z[n_1,n_2]$.

Solution:

$$\begin{aligned} y[n_1,n_2] &= \sum_{m_2} x[n_1,n_2-m_2]a_2[m_2] \\ &= x[n_1,n_2] + 2x[n_1,n_2-1] + x[n_1,n_2-2] \\ &= \begin{cases} 105 & n_2 = n_1-5 \text{ or } n_1-3 \\ 210 & n_2 = n_1-4 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$