Lecture 5: The ”animal kingdom” of heuristics: Admissible, Consistent, zero, Relaxed, Dominant

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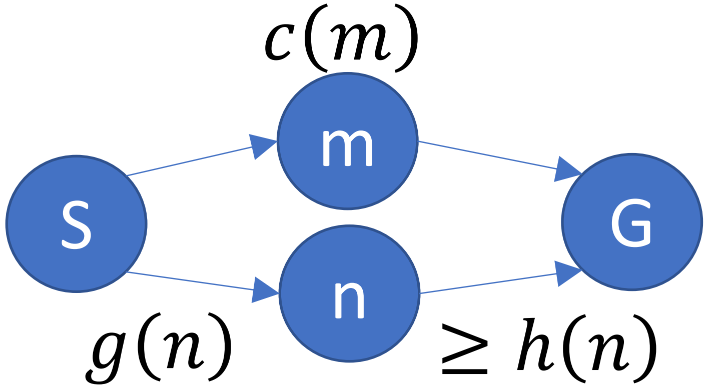
With some slides by Svetlana Lazebnik, 9/2016

Outline of lecture

1. Admissible heuristics
2. Consistent heuristics
3. The zero heuristic: Dijkstra’s algorithm
4. Relaxed heuristics
5. Dominant heuristics

A* Search

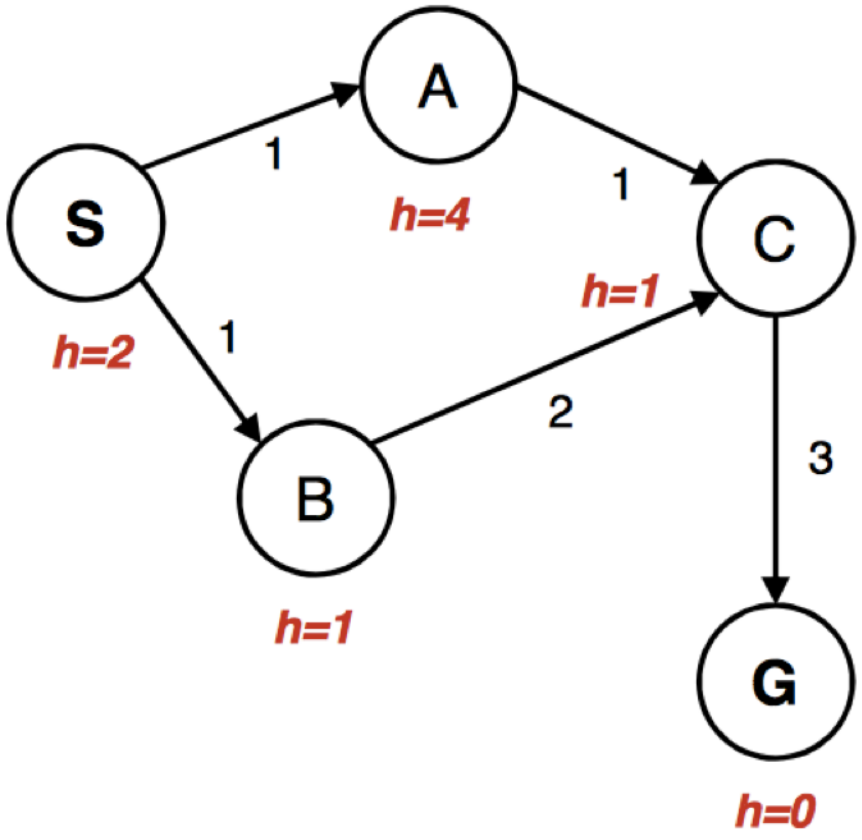


Definition: A* SEARCH

- If $h(n)$ is **admissible** ($d(n) \geq h(n)$), and
- if the frontier is a priority queue sorted according to $g(n) + h(n)$, then
- the FIRST path to goal uncovered by the tree search, path m, is guaranteed to be the SHORTEST path to goal

  ($h(n) + g(n) \geq c(m)$ for every node n that is not on path m)

Bad interaction between A* and the explored set



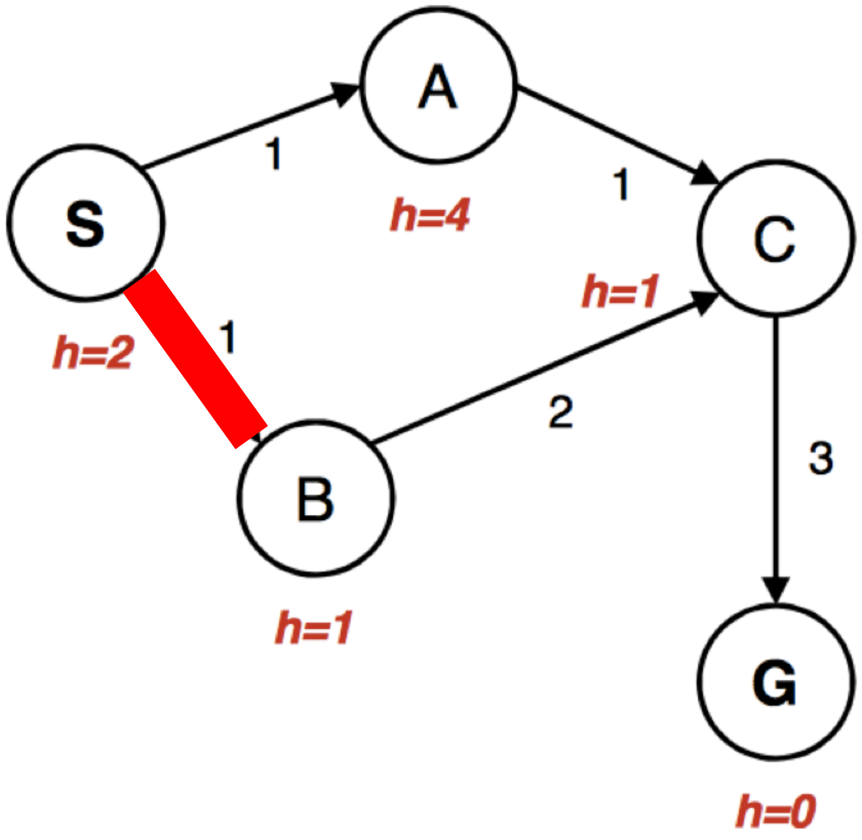
Frontier

S: g(n)+h(n)=2, parent=none

Explored Set

Select from the frontier: S

Bad interaction between A* and the explored set



Frontier

A: g(n)+h(n)=5, parent=S

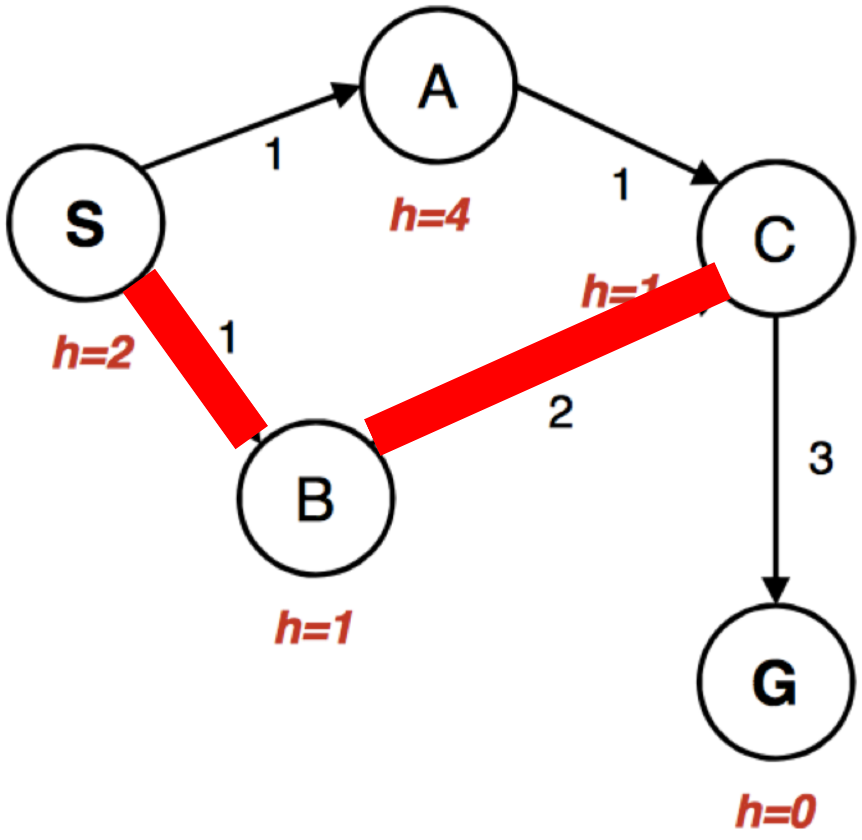
B: g(n)+h(n)=2, parent=S

Explored Set

S

Select from the frontier: B

Bad interaction between A* and the explored set



Frontier

A: g(n)+h(n)=5, parent=S

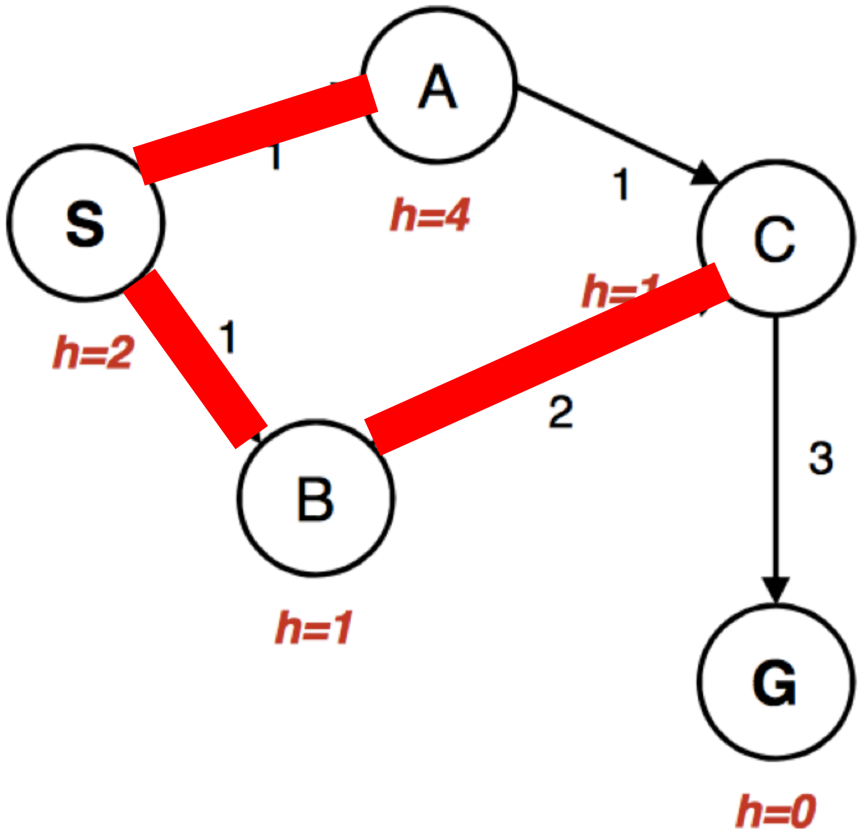
C: g(n)+h(n)=4, parent=B

Explored Set

S, B

Select from the frontier: C

Bad interaction between A* and the explored set



Frontier

A: g(n)+h(n)=5, parent=S

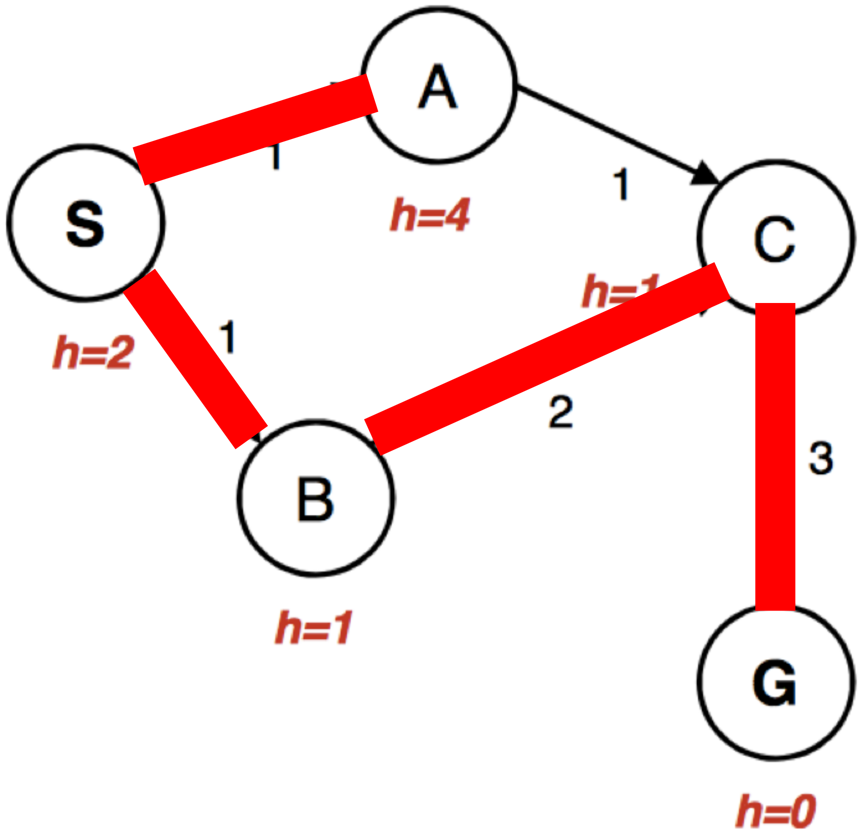
G: g(n)+h(n)=6, parent=C

Explored Set

S, B, C

Select from the frontier: A

Bad interaction between A* and the explored set



Frontier

G: g(n)+h(n)=6, parent=C

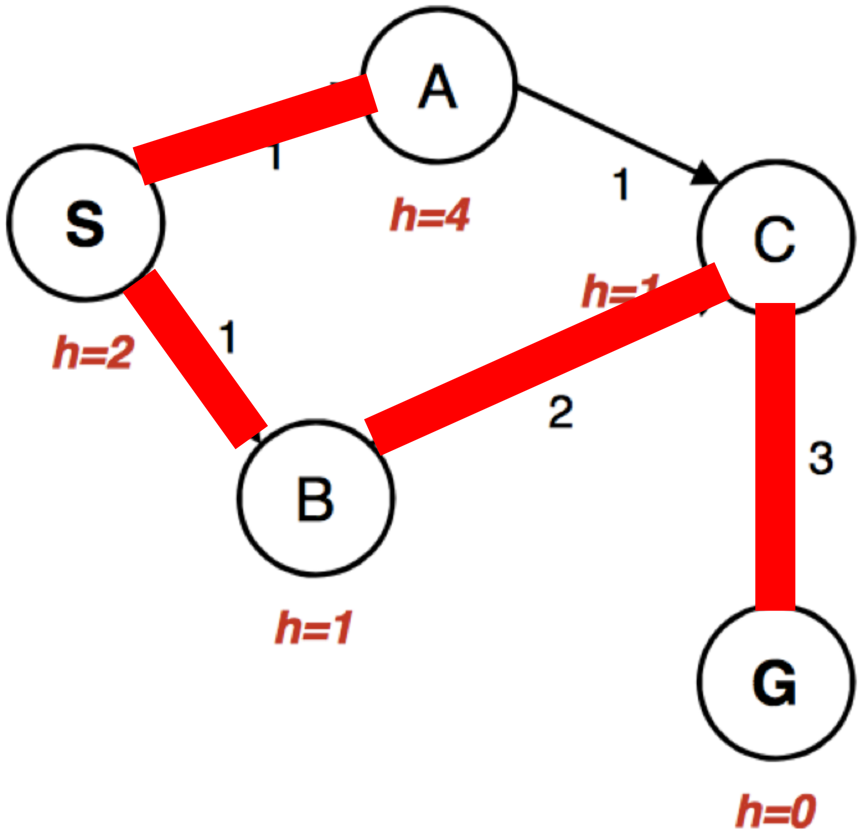
- **Now we would place C in the frontier, with parent=A and h(n)+g(n)=3, except that C was already in the explored set!**

Explored Set

S, B, C

Select from the frontier: **Would be C**, but instead it's G

Bad interaction between A* and the explored set



Return the path S,B,C,G

Path cost = 6

OOPS

Bad interaction between A* and the explored set: Three possible solutions

1. Don't use an explored set
   - This option is OK for any finite state space, as long as you check for loops.
2. Nodes on the explored set are tagged by their h(n)+g(n). If you find a node that's <u>already in the explored set</u>, test to see if the <u>new h(n)+g(n) is smaller than the old one</u>.
   - If so, put the node back on the frontier
   - If not, leave the node off the frontier
3. Use a heuristic that's not only admissible, but also consistent.

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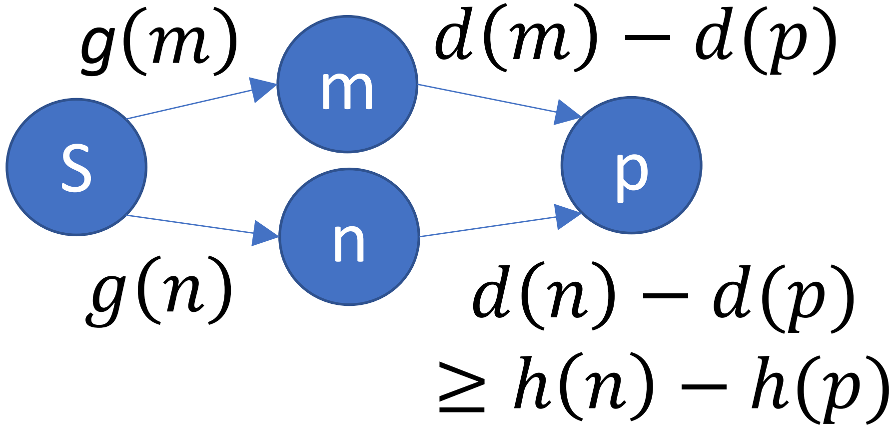
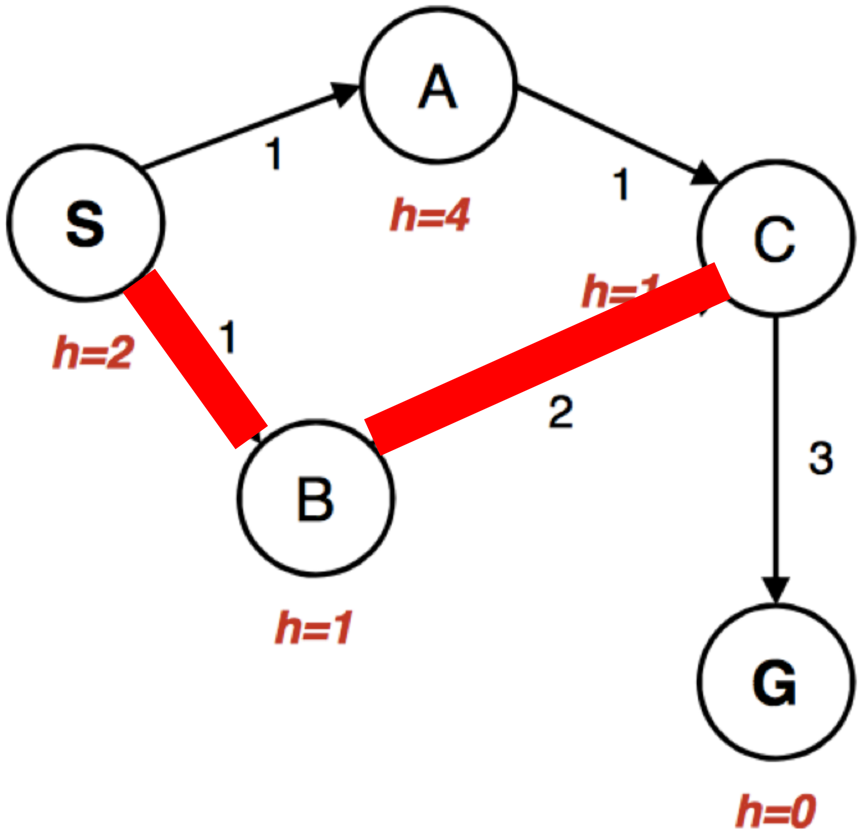
Consistent (monotonic) heuristic



Definition: A **consistent heuristic** is one for which, for every pair of nodes in the graph, $d(n) - d(p) \geq h(n) - h(p)$.

In words: the distance between any pair of nodes is **greater than or equal to** the difference in their heuristics.

A* with an inconsistent heuristic



Frontier

A: g(n)+h(n)=5, parent=S

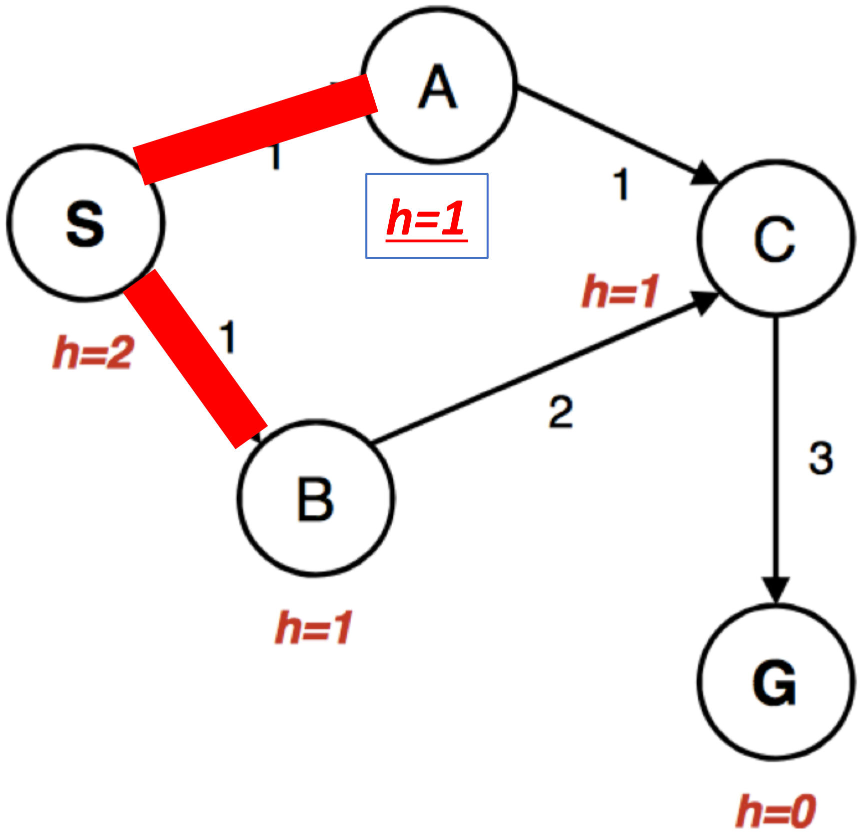
C: g(n)+h(n)=4, parent=B

Explored Set

S, B

Select from the frontier: C

A* with a **consistent** heuristic



Frontier

A: g(n)+h(n)=**2**, parent=S

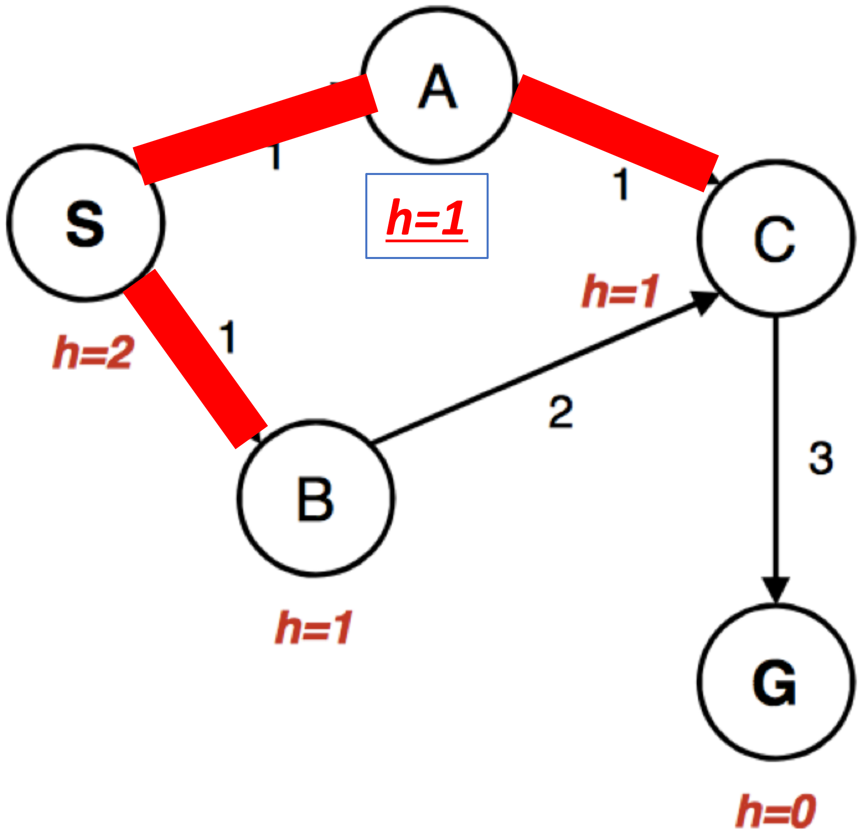
C: g(n)+h(n)=4, parent=B

Explored Set

S, B

Select from the frontier: **A**

A* with a consistent heuristic



Frontier

.

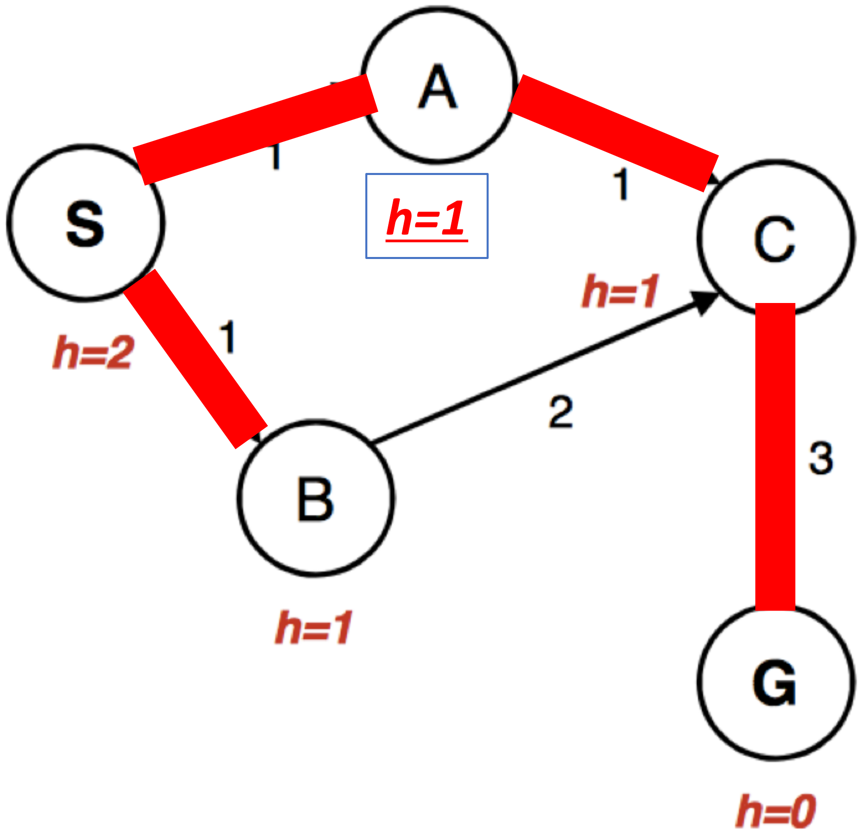
C: g(n)+h(n)=**2**, parent=**A**

Explored Set

S, B, **A**

Select from the frontier: **C**

A* with a **consistent** heuristic



Frontier

.

G: g(n)+h(n)=**5**, parent=C

Explored Set

S, B, **A,** C

Select from the frontier: G

Bad interaction between A* and the explored set: Three possible solutions

1. Don’t use an explored set.

This works for the MP!

2. If you find a node that’s already in the explored set, test to see if the new h(n)+g(n) is smaller than the old one.

Most students find that this is the most computationally efficient solution to the multi-dots problem.

3. Use a consistent heuristic.

Do this too. Consistent: heuristic difference <= actual distance between two nodes. It’s easy to do, because 0 <= d.

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The trivial case: h(n)=0

- A heuristic is **admissible** if and only if $d(n) \geq h(n)$ for every n.
- A heuristic is **consistent** if and only if $d(n, p) \geq h(n) - h(p)$ for every n and p.

- Both criteria are satisfied by $h(n) = 0$.

Dijkstra = A* with h(n)=0

- Suppose we choose $h(n) = 0$
- Then the frontier is a priority queue sorted by
$$g(n) + h(n) = g(n)$$
- In other words, the first node we pull from the queue is the one that's closest to START!! (The one with minimum $g(n)$).
- So this is just Dijkstra's algorithm!

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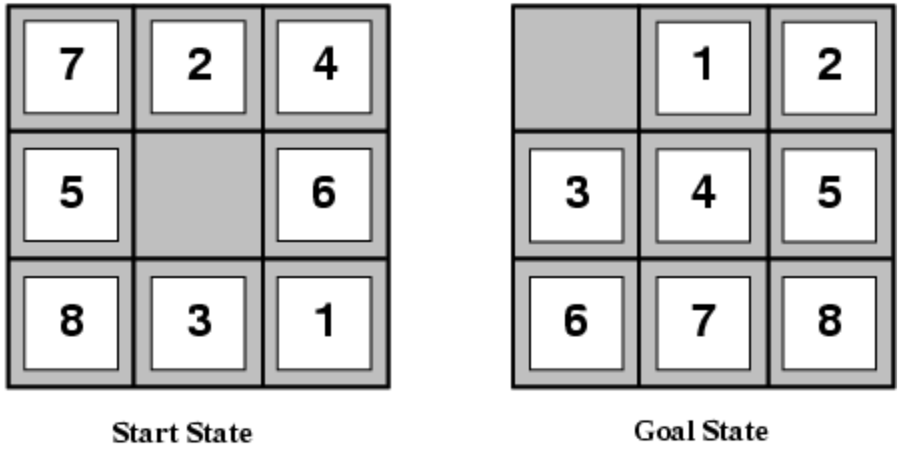
Designing heuristic functions

Now we start to see things that actually resemble the multi-dot problem…

- Heuristics for the 8-puzzle

  $h_1(n)$ = number of misplaced tiles

  $h_2(n)$ = total Manhattan distance (number of squares from desired location of each tile)



$h_1(\text{start}) = 8$

$h_2(\text{start}) = 3+1+2+2+2+3+3+2 = 18$

- Are h_1 and h_2 admissible?

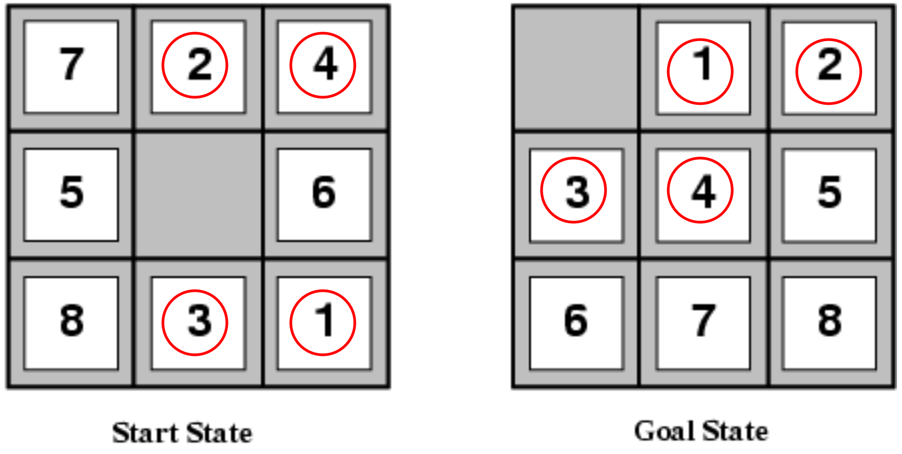
Heuristics from relaxed problems

- A problem with fewer restrictions on the actions is called a relaxed problem
- The cost of an optimal solution to a relaxed problem is an admissible heuristic for the original problem
- If the rules of the 8-puzzle are relaxed so that a tile can move anywhere, then $h_1(n)$ gives the shortest solution
- If the rules are relaxed so that a tile can move to any adjacent square, then $h_2(n)$ gives the shortest solution

Heuristics from subproblems

This is also a trick that many students find useful for the multi-dot problem.

- Let $h_3(n)$ be the cost of getting a subset of tiles (say, 1,2,3,4) into their correct positions
- Can precompute and save the exact solution cost for every possible subproblem instance – *pattern database*
- If the subproblem is O{9^4}, and the full problem is O{9^9}, then you can solve as many as 9^5 subproblems without increasing the complexity of the problem!!



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Dominance

- If h_1 and h_2 are both admissible heuristics and $h_2(n) \geq h_1(n)$ for all n, (both admissible) then h_2 dominates h_1
- Which one is better for search?
  - A* search expands every node with $f(n) < C^*$ or $h(n) < C^* - g(n)$
  - Therefore, A* search with h_1 will expand more nodes = h_1 is more computationally expensive.

Dominance

- Typical search costs for the 8-puzzle (average number of nodes expanded for different solution depths):

- *d*=12 BFS expands 3,644,035 nodes
  $A^*(h_1)$ expands 227 nodes
  $A^*(h_2)$ expands 73 nodes

- *d*=24 BFS expands 54,000,000,000 nodes
  $A^*(h_1)$ expands 39,135 nodes
  $A^*(h_2)$ expands 1,641 nodes

Combining heuristics

- Suppose we have a collection of admissible heuristics $h_1(n)$, $h_2(n)$, …, $h_m(n)$, but none of them dominates the others
- How can we combine them?

$$h(n) = \max\{h_1(n), h_2(n), \ldots, h_m(n)\}$$

All search strategies. C*=cost of best path.

Algorithm	Complete?	Optimal?	Time complexity	Space complexity	Implement the Frontier as a…
BFS	Yes	If all step costs are equal	O(b^d)	O(b^d)	Queue
DFS	No	No	O(b^m)	O(bm)	Stack
UCS	Yes	Yes	Number of nodes w/ g(n) ≤ C*	Number of nodes w/ g(n) ≤ C*	Priority Queue sorted by g(n)
Greedy	No	No	Worst case: O(b^m) Best case: O(bd)	Worse case: O(b^m) Best case: O(bd)	Priority Queue sorted by h(n)
A*	Yes	Yes	Number of nodes w/ g(n)+h(n) ≤ C*	Number of nodes w/ g(n)+h(n) ≤ C*	Priority Queue sorted by h(n)+g(n)